Expansions

In **expansion**, we take a small expression and make it larger, or "expand" it. For example:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Deriving Identities

Identities are derived through multiplication. For example,

$$(a+b)^2$$

means

$$(a+b) \times (a+b)$$

. You can multiply this out:

1. Multiply a by (a + b):

$$a^2 + ab$$

2. Multiply b by (a + b):

$$ab + b^2$$

3. Add them together:

$$a^2 + 2ab + b^2$$

If you forget an identity, you can re-derive it using multiplication.

ICSE Syllabus

The lecture focuses on the following four identities from the syllabus:

- $$(a \pm b)^2$$
- $$(x + a)(x + b)$$

  along with its four sub-identities
- $$(a + b + c)^2$$
- Cubes identities

These identities are also useful in factorization.

The

$$(a + b)^2$$

Identity

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Example Problem

Expand

$$(2x + 7y)^2$$

using the standard formula.

Here,

$$a = 2x$$

and

$$b = 7y$$

.

1. Apply the formula:

$$(2x + 7y)^2 = (2x)^2 + (7y)^2 + 2 \times (2x) \times (7y)$$

2. Simplify:

$$4x^2 + 49y^2 + 28xy$$

Expansion is about turning something small into something larger.

Expanding Numbers

Use

$$(a + b)^2$$

to find

$$101^2$$

.

1. Break down 101 into easy-to-use numbers:

$$101 = 100 + 1$$

2. Apply the formula where

$$a = 100$$

and

$$b = 1$$

:

$$(100 + 1)^2 = 100^2 + 1^2 + 2 \times 100 \times 1$$

3. Solve:

$$10000 + 1 + 200 = 10201$$

Memorizing Identities

Write the identity every time you practice a question. With repetition, the identity will become easier to remember.

The

$$(a-b)^2$$

Identity

$$(a-b)^2 = a^2 + b^2 - 2ab$$

Identities and Simplifications

Using the $(a-b)^2$ Identity

- The identity for $(a-b)^2$ is:

  $a^2 + b^2 - 2ab$

- **Example:** Evaluate $(3x - \frac{1}{2x})^2$
  1. Apply the identity: $(3x)^2 + (\frac{1}{2x})^2 - 2 \times 3x \times \frac{1}{2x}$
  2. Simplify: $9x^2 + \frac{1}{4x^2} - 3$
  3. Note: The x terms cancel out in the $2ab$ part of the equation.

Simplifying Numbers Using Identities

- **Strategy**: Break down numbers into simpler forms using identities.
- **Example**:

  Evaluate 99^2

  1. Rewrite 99 as $100 - 1$.
  2. Apply the $(a-b)^2$ identity:

     $(100-1)^2 = 100^2 + 1^2 - 2 \times 100 \times 1$
  3. Simplify:

     $10,000 + 1 - 200 = 9801$

The $(x+a)(x+b)$ Identity

- The general form is:

  $x^2 + (a+b)x + ab$
- This is the only identity you need to memorize, as it can be manipulated to fit other forms.

Applying the $(x+a)(x+b)$ Identity

- **Example:**

  Evaluate $(x+3)(x+5)$

  1. Use the $(x+a)(x+b)$ identity:

     $x^2 + (3+5)x + (3 \times 5)$
  2. Simplify:

     $x^2 + 8x + 15$

Dealing with Negative Signs

- **Strategy:** Convert subtraction to addition of a negative number.
- **Example:** Evaluate $(x-4)(x+2)$

  1. Rewrite $(x-4)$ as $(x+(-4))$.
  2. Apply the $(x+a)(x+b)$ identity:

     $$x^2+((-4)+2)x+((-4)\times 2)$$

  3. Simplify:

     $$x^2-2x-8$$

The $(a+b+c)^2$ Identity

- The identity is:

  $$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$$

- Only memorize this form and adjust for negative signs as needed.

Using the $(a+b+c)^2$ Identity with Negative Terms

- **Strategy:** Convert negative terms into addition of negative numbers.
- **Example:** Evaluate $(x-2y-z)^2$

  1. Rewrite as $(x+(-2y)+(-z))^2$.
  2. Apply the $(a+b+c)^2$ identity:

     $$x^2+(-2y)^2+(-z)^2+2(x)(-2y)+2(-2y)(-z)+2(-z)(x)$$

  3. Simplify:

     $$x^2+4y^2+z^2-4xy+4yz-2zx$$

- This method avoids memorizing multiple variations of the identity.

Example for Homework

- Evaluate $(2x - 3y + 4z)^2$.
- You can either use the $(a - b + c)^2$ identity directly, or convert it to $(a + b + c)^2$ form by including negative signs.

Cubes Without Direct Calculation

Problem Setup

- Calculate $27^3 - 17^3 - 10^3$ without direct computation.
- Key idea: Note when $a + b + c = 0$.

Important Note

- When $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.
- This is a crucial point to remember when dealing with such problems.

Special Case: $a + b + c = 0$

When you encounter questions where $a + b + c = 0$, you can directly calculate the cube using a simple formula:

- If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

For example, if you have $27^3 + (-17)^3 + (-10)^3$, check if $27 + (-17) + (-10) = 0$.

Since $27 - 17 - 10 = 0$, the expression simplifies to:

$3 \times a \times b \times c = 3 \times 27 \times (-17) \times (-10)$.

This avoids computing large cubes individually.

Expansion of $(a + b)^3$

Key Identities

Remember these two key identities:

- $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
- $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

The second identity is often easier to remember because all the signs are negative.

Example 1: $(x+2)^3$

To expand $(x+2)^3$, use the $(a+b)^3$ formula:

$(x+2)^3 = x^3 + 2^3 + 3 \times x \times 2 \times (x+2)$

$= x^3 + 8 + 6x(x+2)$

$= x^3 + 8 + 6x^2 + 12x$

So, the final expanded form is $x^3 + 6x^2 + 12x + 8$.

Example 2: $(2a-b)^3$

To expand $(2a-b)^3$, use the $(a-b)^3$ formula:

$(2a-b)^3 = (2a)^3 - b^3 - 3 \times 2a \times b \times (2a-b)$

$= 8a^3 - b^3 - 6ab(2a-b)$

$= 8a^3 - b^3 - 12a^2b + 6ab^2$

Thus, the final expanded form is $8a^3 - b^3 - 12a^2b + 6ab^2$.

Combining $(a+b)^2$ and $(a-b)^2$

Simplifying Expressions

Consider the expression $(a+b)^2 + (a-b)^2$:

$(a+b)^2 = a^2 + b^2 + 2ab$

$(a-b)^2 = a^2 + b^2 - 2ab$

Adding them together:

$(a+b)^2 + (a-b)^2 = a^2 + b^2 + 2ab + a^2 + b^2 - 2ab$

The $+2ab$ and $-2ab$ terms cancel out, leaving:

$2a^2 + 2b^2 = 2(a^2 + b^2)$.

Example: $(a + \frac{1}{a})^2 + (a - \frac{1}{a})^2$

Expand $(a + \frac{1}{a})^2 + (a - \frac{1}{a})^2$ using the same approach:

$(a + \frac{1}{a})^2 = a^2 + (\frac{1}{a})^2 + 2 \times a \times \frac{1}{a} = a^2 + \frac{1}{a^2} + 2$

$(a - \frac{1}{a})^2 = a^2 + (\frac{1}{a})^2 - 2 \times a \times \frac{1}{a} = a^2 + \frac{1}{a^2} - 2$

Adding them together:

$a^2 + \frac{1}{a^2} + 2 + a^2 + \frac{1}{a^2} - 2$

The $+2$ and -2 terms cancel out, so:

$2a^2 + \frac{2}{a^2} = 2(a^2 + \frac{1}{a^2})$.

Problem Solving with Given Values

Example: Finding a Value

If $x + y = 4$, find the value of $x^3 + y^3 + 12xy - 64$.

We know $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$.

Given $x + y = 4$, then $(x + y)^3 = 4^3 = 64$.

So, $x^3 + y^3 + 3xy(4) = 64$.

$x^3 + y^3 + 12xy = 64$.

Rearranging gives $x^3 + y^3 + 12xy - 64 = 0$.

Applying Identities to Simplify Expressions

Simplify $\frac{86^3+14^3}{86^2-86\times14+14^2}$.

Recognize this as $\frac{a^3+b^3}{a^2-ab+b^2}$, where $a = 86$ and $b = 14$.

The formula for $a^3 + b^3$ is $(a+b)(a^2 - ab + b^2)$.

So, $\frac{a^3+b^3}{a^2-ab+b^2} = \frac{(a+b)(a^2-ab+b^2)}{a^2-ab+b^2} = a + b$.

Therefore, the expression simplifies to $86 + 14 = 100$.

Using $a^2 - b^2$ Identity

Applying the Formula

The identity $a^2 - b^2 = (a+b)(a-b)$ is widely applicable.

Example 1: $(7p)^2 - (9q)^2$

$(7p)^2 - (9q)^2 = 49p^2 - 81q^2$.

Example 2: $(2x + 3y)(2x - 3y)$

$(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2 = 4x^2 - 9y^2$.

Simplifying Expressions

- When faced with seemingly complex expressions, identify common numbers or expressions to simplify the problem.
- For example, given an expression with terms like $y + 3$ and $y - 3$, and another term $2x - y$ appearing in multiple places, treat $2x - y$ as a single entity for simplification.

Example: Applying Identities

- Use the identity $(a+b)(a-b) = a^2 - b^2$ to simplify expressions.

- Apply the identity $(a-b)^2 = a^2 + b^2 - 2ab$.

- **Example:** Simplify $(2x - y + 3)(2x - y - 3)$

  1. Recognize $a = (2x - y)$ and $b = 3$.
  2. Apply the difference of squares: $(2x - y)^2 - 3^2$.
  3. Expand $(2x - y)^2$ using the $(a-b)^2$ identity: $(2x)^2 + y^2 - 2 \times 2x \times y$.
  4. Simplify to $4x^2 + y^2 - 4xy - 9$.
  5. Further simplification yields $16x^2 + y^2 - 4xy - 9$.

Finding Coefficients

- To find the coefficient of a term in a polynomial, first, **expand the expression** by multiplying all terms.
- The **coefficient** is the number associated with a variable in a term.

Example: Multiplying Polynomials

- **Problem:** Find the coefficient of x in the expansion of $(x-3)(x+7)(x-4)$.

    1. Multiply $(x-3)(x+7)$ first:
        - $x \times x = x^2$
        - $x \times 7 = 7x$
        - $-3 \times x = -3x$
        - $-3 \times 7 = -21$
        - Result: $x^2 + 4x - 21$
    2. Multiply the result by $(x-4)$:
        - $x^2 \times x = x^3$
        - $x^2 \times -4 = -4x^2$
        - $4x \times x = 4x^2$
        - $4x \times -4 = -16x$
        - $-21 \times x = -21x$
        - $-21 \times -4 = 84$
        - Result: $x^3 - 4x^2 + 4x^2 - 16x - 21x + 84$
    3. Simplify: $x^3 - 37x + 84$
    4. The coefficient of x is -37.

Solving Algebraic Problems

- Sometimes, problems require you to manipulate given equations to find a specific value.
- Squaring both sides of an equation can reveal new relationships between variables.

Example: Finding $x^2 + y^2$

- **Problem:** If $x - y = 8$ and $xy = 5$, find the value of $x^2 + y^2$.

  1. Square both sides of the equation $x - y = 8$:
     - $(x - y)^2 = 8^2$
     - $x^2 + y^2 - 2xy = 64$
  2. Substitute the value of $xy = 5$:
     - $x^2 + y^2 - 2 \times 5 = 64$
     - $x^2 + y^2 - 10 = 64$
  3. Solve for $x^2 + y^2$:
     - $x^2 + y^2 = 64 + 10$
     - $x^2 + y^2 = 74$

Example: Finding $a + b$ and $a - b$

- **Problem**: If $a^2 + b^2 = 13$ and $ab = 6$, find $a + b$ and $a - b$.

  1. Find $a + b$:
     - $(a + b)^2 = a^2 + b^2 + 2ab$
     - $(a + b)^2 = 13 + 2 \times 6 = 13 + 12 = 25$
     - $a + b = \pm\sqrt{25} = \pm 5$
  2. Find $a - b$:
     - $(a - b)^2 = a^2 + b^2 - 2ab$
     - $(a - b)^2 = 13 - 2 \times 6 = 13 - 12 = 1$
     - $a - b = \pm\sqrt{1} = \pm 1$

Example: Using Identities and Given Values

- Problem: Given $a + b = 4$ and $ab = -12$, find $a - b$ and $a^2 - b^2$.

    1. Find $a - b$:
        - First, find $a^2 + b^2$ using $(a + b)^2 = a^2 + b^2 + 2ab$:
            - $4^2 = a^2 + b^2 + 2 \times (-12)$
            - $16 = a^2 + b^2 - 24$
            - $a^2 + b^2 = 16 + 24 = 40$
        - Now, use $(a - b)^2 = a^2 + b^2 - 2ab$:
            - $(a - b)^2 = 40 - 2 \times (-12) = 40 + 24 = 64$
            - $a - b = \pm\sqrt{64} = \pm 8$
    2. Find $a^2 - b^2$:
        - $a^2 - b^2 = (a + b)(a - b)$
        - $a^2 - b^2 = 4 \times (\pm 8) = \pm 32$

Algebraic Identities and Problem Solving

Calculating $a^2 - b^2$

- In one instance, we have $+8$ and -8.
- $8 \times 4 = 32$
- $8 \times -4 = -32$
- Therefore, $a^2 - b^2 = \pm 32$

Solving for $x^2 + \frac{1}{x^2}$ given $x - 3 = \frac{1}{x}$

1. Shift the equation to get $x - \frac{1}{x} = 3$.
2. Square both sides: $(x - \frac{1}{x})^2 = 3^2$.
3. Expand: $x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x} = 9$.
4. Simplify: $x^2 + \frac{1}{x^2} - 2 = 9$.
5. Solve for $x^2 + \frac{1}{x^2}$: $x^2 + \frac{1}{x^2} = 11$.

Finding $x + y$ and $x - y$

Given:

- $x^2 + y^2 = 34$
- $xy = 10\frac{1}{2} = \frac{21}{2}$

Goal: Find the value of $2(x+y)^2 + (x-y)^2$

1. Recall the identities:
    - $(a+b)^2 = a^2 + b^2 + 2ab$
    - $(a-b)^2 = a^2 + b^2 - 2ab$
2. Find $(x+y)^2$:
    - $(x+y)^2 = x^2 + y^2 + 2xy$
    - $(x+y)^2 = 34 + 2 \times \frac{21}{2}$
    - $(x+y)^2 = 34 + 21 = 55$
3. Find $(x-y)^2$:
    - $(x-y)^2 = x^2 + y^2 - 2xy$
    - $(x-y)^2 = 34 - 2 \times \frac{21}{2}$
    - $(x-y)^2 = 34 - 21 = 13$
4. Substitute into the expression:
    - $2(x+y)^2 + (x-y)^2 = 2(55) + 13$
    - $= 110 + 13 = 123$

Solving for $8a^3 - 27b^3$ using Cubing

Given:

- $2a - 3b = 3$
- $ab = 2$

1. Cube both sides of the first equation: $(2a-3b)^3 = 3^3$.
2. Expand using the identity: $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$.
    - $(2a)^3 - (3b)^3 - 3(2a)(3b)(2a-3b) = 27$
    - $8a^3 - 27b^3 - 18ab(2a-3b) = 27$
3. Substitute the given values:
    - $8a^3 - 27b^3 - 18(2)(3) = 27$
    - $8a^3 - 27b^3 - 108 = 27$
4. Solve for $8a^3 - 27b^3$:
    - $8a^3 - 27b^3 = 27 + 108$
    - $8a^3 - 27b^3 = 135$

Word Problem: Sum and Product of Two Numbers

Problem: The sum and product of two numbers are 8 and 15 respectively. Find the sum of their cubes.

1. **Let** the two numbers be x and y.
2. **Given**:
    - $x + y = 8$
    - $xy = 15$
3. **Cube** both sides of $x + y = 8$: $(x + y)^3 = 8^3$.
4. **Expand** using the identity: $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$.
    - $x^3 + y^3 + 3(15)(8) = 512$
    - $x^3 + y^3 + 360 = 512$
5. **Solve** for $x^3 + y^3$:
    - $x^3 + y^3 = 512 - 360$
    - $x^3 + y^3 = 152$

Homework Question

Solve a similar type of question as practice. (Question not provided in the text, so cannot be included here).